Journal of Statistical Physics, Vol. 124, No. 6, September 2006 (© 2006)
DOI: 10.1007/s10955-006-9179-7

Exact Analysis of Level-Crossing Statistics for ($d + 1$)-Dimensional Fluctuating Surfaces

A. Bahraminasab,[1,2] M. Sadegh Movahed,[2,3] S. D. Nasiri,[4] A. A. Masoudi[5] and Muhammad Sahimi[6]

Received April 30, 2006; accepted July 9, 2006
Published Online: September 15, 2006

We carry out an exact analysis of the average frequency $\nu^+_{\alpha x_i}$ in the direction x_i of positiveslope crossing of a given level α such that, $h(\mathbf{x}, t) - \bar{h} = \alpha$, of growing surfaces in spatial dimension d. Here, $h(\mathbf{x}, t)$ is the surface height at time t, and $\bar{h}$ is its mean value. We analyze the problem when the surface growth dynamics is governed by the Kardar-Parisi-Zhang (KPZ) equation without surface tension, in the time regime prior to appearance of cusp singularities (sharp valleys), as well as in the random deposition (RD) model. The total number N^+ of such level-crossings with positive slope in all the directions is then shown to scale with time as $t^{d/2}$ for both the KPZ equation and the RD model.

KEY WORDS: level crossing analysis, surface growth

PACS number(s): 52.75.Rx, 68.35.Ct

1. INTRODUCTION

Due to their practical applications and fundamental interest, a great amount of effort has been devoted to understanding the mechanism(s) of growth of thin films, and the kinetic roughening of their surface during the growth. Analytical as well as numerical analyses of many models of surface growth suggest, quite generally, that certain properties of the surface exhibit dynamic scaling.[(1–6)] In

[1] International Center for Theoretical Physics, Strada Costiera 11, I-34100 Trieste, Italy.
[2] Deptartment of Physics, Sharif University of Technology, Tehran 11365-9161, Iran.
[3] Institute for Studies in Theoretical Physics and Mathematics, Tehran 19395-5531, Iran.
[4] Shahid Beheshti University, Tehran 19395-4716, Iran.
[5] Department of Physics, Alzahra University, Tehran 19834, Iran.
[6] Mork Family Department of Chemical Engineering & Materials Science, University of Southern California, Los Angeles, California 90089-1211, USA.

order to derive quantitative information about the films' surface morphology, one considers a sample of size L and defines the mean height $\bar{h}$ of the growing film and its surface width w by,[1]

$$\bar{h}(L,t) = \frac{1}{L}\int_{-L/2}^{L/2} d\mathbf{x}\, h(\mathbf{x},t), \tag{1}$$

where $h(\mathbf{x}, t)$ is the surface height, and

$$w(L,t) = [\langle (h - \bar{h})^2 \rangle]^{1/2}, \tag{2}$$

with $\langle\cdot\rangle$ indicating an average over different realizations of the surface. Starting from a flat surface as the initial condition, it was proposed by Family and Vicsek[7] that rescaling the space and time variables by, respectively, b and b^z rescales the surface width w by b^χ, $w(bL, b^z t) = b^\chi w(L,t)$, implying that

$$w(L,t) = L^\chi f(t/L^z), \tag{3}$$

where $f(x)$ is a scaling function, and z is the dynamic exponent. If for large t and fixed $L(t/L^z \to \infty)$ the width w saturates, then one must have, $f(x) \to$ constant as $x \to \infty$. However, for fixed and large L and $1 \ll t \ll L^z$, one expects the correlations in the height fluctuations to grow within only a distance $t^{1/z}$ and, thus, w must be independent of L. This implies that for $x \ll 1$, $f(x) \sim x^\beta$ with, $\beta = \chi/z$. Therefore, the dynamic scaling of Family and Vicsek[7] postulates that, $w(L,t) \sim t^\beta$ for $1 \ll t \ll L^z$, and $w \sim L^\chi$ for $t \gg L^z$. The roughness exponent χ and the dynamic exponent z characterize the self-affine geometry of the surface and its dynamics, respectively. These scaling relations have been tested for a variety of growing surfaces by extensive numerical simulations and analytical calculations.

In this paper we introduce the concept of level crossing in the context of surface growth processes. In the level-crossing analysis one is interested in determining the average frequency, $\nu^+_{\alpha x_i}$, in the x_i-direction of observing a given level α for the function, $h - \bar{h} = \alpha$, in the growing film. We show that $\nu^+_{\alpha x_i}$ is written in terms of the joint probability distribution function (PDF) of $h - \bar{h}$ and the gradient of $h(\mathbf{x}, t)$. Therefore, $\nu^+_{\alpha x_i}$ carries the same information about the film's surface which is contained in the joint PDF of the fluctuations in the height $h(\mathbf{x}, t)$ and its gradient. Our goal in this paper is to study and analyze the quantity $\nu^+_{\alpha x_i}$ at time t for a growing surface in a sample of size L.

In addition, we introduce an integrated quantity, $N^+_{x_i}$, defined by, $N^+_{x_i}, = \int_{-\infty}^{\infty} d\alpha \nu^+_{\alpha x_i}$ which measures the total number of positive-slope crossings of the surface in the x_i-direction, and is expected to become size-dependent in the stationary state. We determine exactly the time- and height-dependence of $\nu^+_{\alpha x_i}$ and $N^+_{x_i}$ for the Kardar-Parisi-Zhang (KPZ) equation in the limit of strong coupling, over time scales prior to the emergence of cusp singularities (sharp valleys). We also derive exact expressions for the same quantities for the random deposition model.

The statistics of level-crossings that we study in the present paper are closely related to the concepts of first passage and persistence of fluctuating interfaces in both space and time, which have recently been studied in many papers.(8,9) In particular, the concept of spatial persistence was introduced by Majumdar and Bray(10) (MB), who showed that the probability $P_0(l)$ that the height of a fluctuating $(d+1)$–dimensional surface in its steady state stays above its initial value up to a distance l, along any linear cut in the d–dimensional space, decays as, $P_0(l) \sim l^{-\theta}$ Here, θ is a spatial persistence exponent which takes on distinct values, depending on how the point from which the distance l is measured is specified. Their analysis was carried out for a fluctuating interface at *steady state.* The exponents were shown to map onto the corresponding temporal persistence exponents for a generalized $(d+1)$–dimensional random walk.

There are a few important differences between our work and that of MB. First, all the analysis and calculations that we present in the present paper are valid at times $t < t^*$, where t^* is the time scale over which sharp valleys (cusp singularities) are developed in the growing surface. Therefore, the time regime that we investigate in the present paper is completely different from what was considered by MB, as we study the probability of crossing a certain height with a positive slope at times $t < t^*$, for which we study the *short-time* behavior of the KPZ equation. Secondly, we study the KPZ model in the limit of zero surface tension – a nonlinear equation – and derive our exact results for any spatial dimension d, whereas MB derived their results for a linear model (the Gaussian model), except when they considered the KPZ equation in $(1+1)$–dimensions. Thirdly, we study the problem assuming a flat initial surface, which is distinct from "finite initial starting point" of MB (see below).

The plan of this paper is as follows. In Sec. 2 we discuss the connection between $\nu^+_{\alpha x_i}$ and the underlying PDF of growing surfaces. In Sec. 3 we derive an integral representation for $\nu^+_{\alpha x_i}$ for the KPZ equation in $(d+1)$–dimensions in the strong coupling limit, before the emergence of the cusp singularities. Section 4 presents our exact results for the random deposition model. The paper is summarized in Sec. 5, while four Appendices provide the details of the derivation of our results.

2. LEVEL-CROSSING IN GROWING SURFACES

Consider a sample of an ensemble of functions which make up the homogeneous random process $h(\mathbf{x}, t)$, representing the height of a growing surface. Let n^+_α denote the number of positive-slope crossings such that, $h(\mathbf{x}, t) - \bar{h} = \alpha$. In a time t and for a typical growing surface of linear size L (see Fig. 1), let the mean value of n^+_α for all the samples be $N^+_\alpha(L)$,

$$N^+_\alpha(L) = E[n^+_\alpha(L)], \tag{4}$$

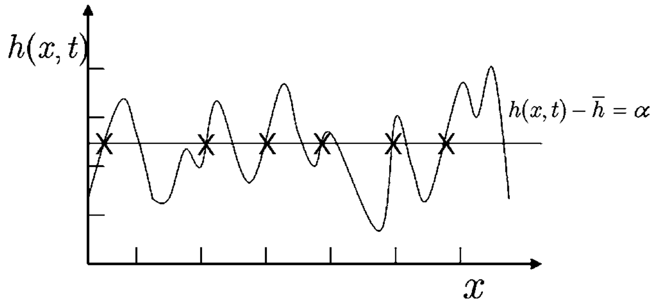

Fig. 1. Positive-slope crossing of the level $h(x, t) - \bar{h} = \alpha$.

where E denotes the expectation or mean value of the quantity. If we consider N_α^+ in a second segment of size L immediately following the first, then, since the process is homogeneous, we obtain the same result as given by Eq. (4). Thus, for the two intervals together we obtain,

$$N_\alpha^+(2L) = 2N_\alpha^+(L), \tag{5}$$

from which it follows that, for a homogeneous process, the average number of crossings is proportional to the space interval L. Hence,

$$N_\alpha^+(L) \propto L, \tag{6}$$

or

$$N_\alpha^+(L) = \nu_\alpha^+ L. \tag{7}$$

We now consider how ν_α^+ is deduced from the underlying probability distribution for $h(\mathbf{x}, t) - \bar{h}$. Consider a small length dl of a typical sample function. Since we assume that the process $h(\mathbf{x}, t) - \bar{h}$ is a smooth function of $\mathbf{x}$ with no sudden increase or decrease, and that dl is small enough, then the sample can only cross the level $h(\mathbf{x}, t) - \bar{h} = \alpha$ with a positive slope, if $h(\mathbf{x}, t) - \bar{h} < \alpha$ at the beginning of the interval dl. Furthermore, there is a minimum slope at position $\mathbf{x}$ if the level $h(\mathbf{x}, t) - \bar{h} = \alpha$ is to be crossed in the interval dl, depending on the value of $h(\mathbf{x}, t) - \bar{h}$ at $\mathbf{x}$. Therefore, there will be a positive crossing of $h(\mathbf{x}, t) - \bar{h} = \alpha$ in the next space interval dl, if at position $\mathbf{x}$,

$$h(\mathbf{x}, t) - \bar{h} < \alpha, \text{ and } \quad \frac{d(h - \bar{h})}{dl} > \frac{\alpha - [h(\mathbf{x}, t) - \bar{h}]}{dl}. \tag{8}$$

If the above conditions are satisfied, then there will be a high probability of crossing the level in the interval dl.[11,12]

In order to determine whether conditions (8) are satisfied at an arbitrary location $\mathbf{x}$, we must determine how values of $y = h(\mathbf{x}, t) - \bar{h}$ and $y' = dy/dl$ are distributed by considering their joint probability density $P(y, y')$. Suppose that the level $y = \alpha$ and the interval dl are specified. Then, we are interested only in those values of y such that $y < \alpha$ and of $y' = dy/dl > (\alpha - y)/dl$, which represent the region between the lines $y = \alpha$ and $y' = (\alpha - y)/dl$ in the plane (y, y'). Hence, the probability of a positive-slope crossing of $y = \alpha$ in the interval dl is given by,

$$\int_0^\infty dy' \int_{\alpha - y'dl}^{\alpha} dy\ P(y, y'). \tag{9}$$

As $dl \to 0$, one has,

$$P(y, y') = P(y = \alpha, y') \tag{10}$$

Since for large values of y and y' the PDF approaches zero fast enough, Eq. (9) may be written as

$$\int_0^\infty dy' \int_{\alpha - y'dl}^{\alpha} dy\ P(y = \alpha, y'), \tag{11}$$

in which the integrand is no longer a function of y, so that the first integral in Eq. (11) is simply, $\int_{\alpha - y'dl}^{\alpha} dy\ P(y = \alpha, y') = P(y = \alpha, y')y'dl$. Then, the probability of a positive-slope crossing of $y = \alpha$ in the interval dl is equal to

$$dl \int_0^\infty P(\alpha, y')y'\, dy'. \tag{12}$$

Since according to Eq. (7) the average number of positive-slope crossings over a length scale L is $\nu_\alpha^+ L$, then, the average number of crossings in the interval dl is $\nu_\alpha^+ dl$. It then follows that the average number of positive-slope crossings of the level $y = \alpha$ in the interval dl is equal to the probability of positive-slope crossing of the level $y = \alpha$ in dl, which is true only if dl is small and the process $y(x)$ is smooth enough that there cannot be more than one crossing of $y = \alpha$ in the interval dl. In that case, we have $\nu_\alpha^+ dl = dl \int_0^\infty P(\alpha, y')y'\, dy'$ and, therefore,

$$\nu_\alpha^+ = \int_0^\infty P(\alpha, y')y'\, dy'. \tag{13}$$

In the following sections we will derive exact expressions for ν_α^+ via the joint PDF of $h(\mathbf{x}, t) - \bar{h}$ and the height gradient, for both the KPZ and random deposition models. To derive the joint PDF we use the master equation method.[13–16] This approach enables us to determine ν_α^+ in terms of a generating function. For example, the generating function for a (2 + 1)–dimensional surface is given by (see below),

$$z(\lambda, \mu, \mathbf{x}, t) = \langle \exp\{-i\lambda[h(\mathbf{x}, t) - \bar{h}] - i\mu u(\mathbf{x}, t)\}\rangle, \tag{14}$$

where, $u(\mathbf{x}, t) = -\nabla h$.

3. ANALYSIS OF LEVEL-CROSSING FOR THE KPZ EQUATION

In the KPZ model in $(d+1)$–dimensions, the surface height $h(\mathbf{x}, t)$ at position $\mathbf{x}$ on top of the substrate, in the limit of the zero surface tension, satisfies the following stochastic equations

$$\frac{\partial h}{\partial t} = h_t = \frac{1}{2}\bar{\alpha}\sum_{i=1}^{d} u_i^2 + f, \tag{15}$$

$$\frac{\partial u_i}{\partial t} = u_{i,t} = \bar{\alpha}\sum_{j=1}^{d} u_l p_{ji} + f_{x_i}, \tag{16}$$

where $u_i = \partial h/\partial x_i = h_{x_i}$, and $p_{ij} = \partial h_{x_i}/\partial x_j$. Here, f is a zero-mean random force with a Gaussian correlation in space and white noise in time:

$$\langle f(\mathbf{x}, t) f(\mathbf{x}', t')\rangle = 2D_0 D(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{17}$$

where $D(\mathbf{x}-\mathbf{x}')$, an even function of its argument, is the spatial correlation function which takes on the following form,

$$D(\mathbf{x}-\mathbf{x}') = \frac{1}{\pi^{d/2}\sigma_{x_1}\sigma_{x_2}\cdots\sigma_{x_d}} \exp\left[-\sum_{1}^{d}\frac{(x_i - x_i')^2}{\sigma_{x_i}^2}\right], \tag{18}$$

with σ_{x_i} being the standard deviations in the x_i–direction. The parameters $\bar{\alpha}$ and D_0 describe, respectively, lateral growth and the noise strength. Typically, in order to account for short-range correlations, the correlation function is taken to be, $D(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$, but we regularize the delta-function correlation by a Gaussian function. When the standard deviations $\sigma = \sigma_{xi}$ are much smaller than the system's size L, we would expect the model to exhibit short-range correlations in the f term. Therefore, we emphasize that our analysis is for the case when, $\sigma \ll L$.

As analyzed in Appendices A, B, and C, we define the generating function, $Z(\lambda, \mu_i, x_i, t) = \langle\Theta(\lambda, \mu_i, x_i, t)\rangle$, for the fields $\tilde{h} = h(\mathbf{x}, t) - \bar{h}$ and $u_i = h_{x_i}$, where

$$\Theta = \exp\left\{-i\lambda[h(\mathbf{x}, t) - \bar{h}(t)] - i\sum_{i=1}^{d}\mu_i u_i\right\}. \tag{19}$$

λ and μ_i are the sources of $\tilde{h}$ and u_i, respectively, and, $i, j = 1, \ldots, d$. Assuming statistical homogeneity, i.e., assuming that, $\partial Z/\partial\mathbf{x} = 0$, it follows from Eqs. (15) and (16) that Z satisfies the following equation (see Appendix C for details),

$$Z_t = \frac{\partial Z}{\partial t} = i\lambda\gamma(t)Z - \frac{1}{2}i\lambda\bar{\alpha}\sum_{l} Z_{\mu_l\mu_l} - \lambda^2 k(\mathbf{0})Z\sum_{l}\mu_l^2 k''(\mathbf{0})Z. \tag{20}$$

where, $k(\mathbf{x}-\mathbf{x}') = 2D_0 D(\mathbf{x}-\mathbf{x}')$, $\gamma(t) = \bar{h}_t$, $k(\mathbf{0}) = 2D_0/(\pi\sigma^d)$, and $k_{x_i x_i}(\mathbf{0}) = -4D_0/(\pi\sigma^{2+d})$, where we use, $\sigma = \sigma_{x_i} = \sigma_{x_j}$, simplicity.

In trying to develop a statistical theory of level-crossings in rough surfaces, it becomes clear that the interdependence of the statistics for height difference $h(\mathbf{x}, t) - \bar{h}$ and height gradient must be taken into account. The very existence of a nonlinear term in the KPZ equation leads to development of the cusp singularities (sharp valleys) in a *finite time* and in the strong coupling limit, hence forcing one to distinguish between different time regimes. It was shown recently that, starting from a flat surface, the KPZ equation will develop sharp-valley singularities after a time scale t^* where,[13] $t^* \sim D_0^{-1/3}\bar{\alpha}^{-2/3}\sigma^{(d+4)/3}$. This implies that for times $t < t^*$ the relaxation contributions tend to vanish in the strong coupling limit. In this regime one can derive closed-form solution for the generating function. Starting from a flat surface, i.e., from $h(\mathbf{x}, 0) = 0$, and $u(\mathbf{x}, 0) = 0$, one has the following solution (see Appendix C)

$$Z(\lambda, \mu_1, \cdots, \mu_d, t) = F_1(\lambda, \mu_1, t) \cdots F_d(\lambda, \mu_d, t) \exp[-\lambda^2 k(\mathbf{0})t], \tag{21}$$

with

$$F_j(\lambda, \mu_j, t) = \left\{1 - \tanh^2\left[\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda t}\right]\right\}^{-1/4} \times \exp\left\{-\frac{1}{2}i\mu_j^2\sqrt{\frac{2ik_{xx}(\mathbf{0})}{\bar{\alpha}\lambda}}\tanh\left[\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda t}\right] - \frac{1}{2}i\bar{\alpha}k_{xx}(\mathbf{0})\lambda t^2\right\}. \tag{22}$$

To derive a closed expression for the PDF $P(\bar{h}, \mathbf{u}_i, t)$, one needs to determine such moments as $\langle h^n u_i^m u_j^l p_{ij}$. As shown in Appendix B, such moments are identically zero.[14] Using this result, it can then be shown that $P(\tilde{h}, \mathbf{u}_i, t)$ takes on the following expression (see Appendix C),

$$P(\tilde{h}, u_{x_i}, t) = \frac{1}{(2\pi)^{d+1}} \int d\lambda\, d\mu_1 \cdots d\mu_d \exp\left(i\lambda\tilde{h} + i\sum_{i=1}^{d}\mu_i u_i\right) Z(\lambda, \mu_1, \cdots, u_d, t). \tag{23}$$

According to the Eq. (13), the frequency of crossing a definite height $h(\mathbf{x}, t) - \bar{h} = \alpha$ with a positive slope in, for example, the x_1–direction, is given by,

$$\nu_{\alpha x_1}^{+} = \int_0^{\infty} du_{x_1} u_{x_1} \int_{-\infty}^{\infty} du_{x_2} \cdots du_{x_d} P(\alpha, u_{x_1}, \ldots u_{x_1}, t). \tag{24}$$

Using the Eqs. (21)–(23), $\nu_{\alpha x_1}^{+}$ is then given by,

$$\begin{aligned}\nu_{\alpha x_1}^{+} &= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} -\frac{e^{i\lambda\alpha}}{\mu_1^2} Z(\lambda, \mu_1, \mu_2 \to 0, \cdots, \mu_d \to 0, t)\, d\lambda\, d\mu_1, \\ &= \frac{1}{2\pi^{3/2}}\int_{-\infty}^{\infty} d\lambda \exp\left[i\lambda\alpha - \lambda^2 k(\mathbf{0})t - i\bar{\alpha}k_{xx}(\mathbf{0})\lambda t^2\right]\zeta,\end{aligned} \tag{25}$$

where

$$\zeta = \frac{\sqrt{\frac{1}{2} i \sqrt{\frac{2ik_{xx}(\mathbf{0})}{\bar{\alpha}\lambda}} \tanh\left[\mathrm{t}\sqrt{2ik_{\mathrm{xx}}(\mathbf{0})\bar{\alpha}\lambda} \right]}}{\sqrt{1 - \tanh^2\left[t\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda} \right]}}. \tag{26}$$

To compute the integral in Eq. (25), we used the numerical integration software developed by Piessens *et al.*[17] which uses a globally-adaptive scheme based on Gauss-Kronrod quadrature rules. In Fig. 2 we plot $\nu^+_{\alpha x_1}$ for times, $t/t^* = 0.05, 0.1$, and 0.2, which are before the emergence of the cusp singularities. To derive an expression for $N^+_{x_1}$, the total number of level-crossings with positive slopes in the x_1–directions, let us express it in terms of the generating function Z. It can be shown straightforwardly that $N^+_{x_1}$ is written in terms of the generating function Z as

$$N^+_{x_1} = \frac{1}{2\pi} \int_{-\infty}^{\infty} -\frac{1}{\mu_1^2} Z(\lambda \to 0, \mu_1, \mu_2 \to 0, \cdots, \mu_d \to 0, t)\, d\mu_1 = \lim_{\lambda \to 0} \frac{\zeta}{\sqrt{\pi}}. \tag{27}$$

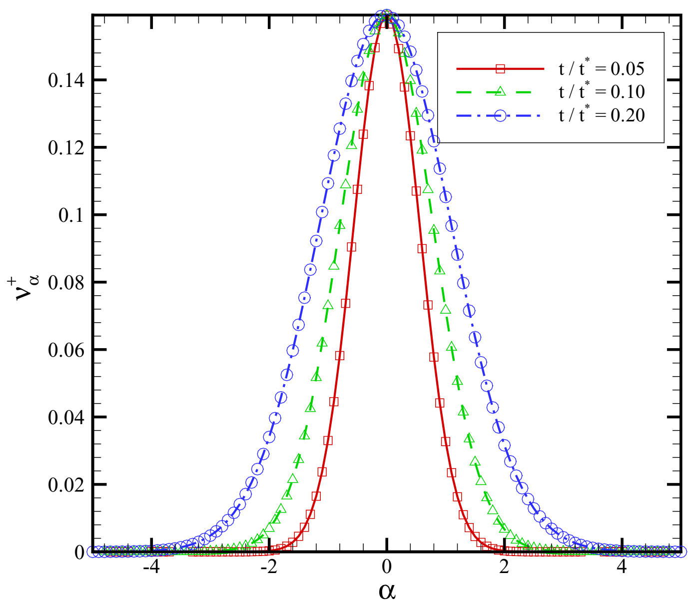

Fig. 2. Plot of ν^+_α vs α for the KPZ equation in the strong coupling and before the emergence of sharp valleys for various times, t/t^* in $(2+1)$–dimensions. (color online)

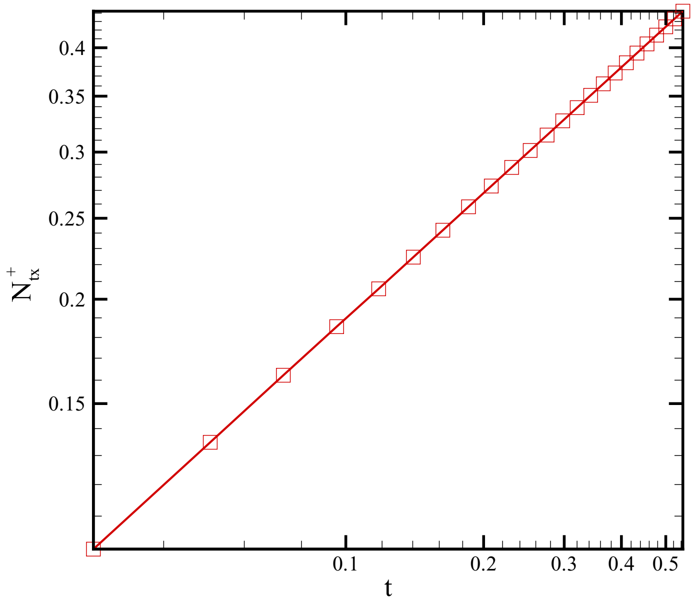

Fig. 3. Logarithmic plot of $N^+_{x_1}$ vs t for the KPZ equation in the strong coupling, and before the emergence of sharp valleys in (2 + 1)–dimensions. (color online)

Using the Eq. (26) one finds that, $N^+_{x_1} \sim t^{1/2}$. In Fig. 3 (obtained by direct numerical integration of $N^+_{x_1} = \int_{-\infty}^{\infty} \nu^+_{\alpha x_1} d\alpha$) we plot $N^+_{x_1}$ vs t, which also indicates that, $N^+_{x1} = \sim t^{1/2}$, in agreement with the analytical prediction.

4. ANALYSIS OF LEVEL-CROSSING FOR THE RANDOM DEPOSITION MODEL

In the random deposition model, the height of each column performs an independent random walk. This model leads to unrealistically rough surfaces, with its width growing with the exponent $\beta = \chi/z = 1/2$ without ever saturating. In the continuum limit the random deposition model is described by,

$$\frac{\partial}{\partial t} h(\mathbf{x}, t) = f(\mathbf{x}, t), \quad \frac{\partial}{\partial t} u_i(\mathbf{x}, t) = f_{x_i}, i = 1, \cdots, d, \tag{28}$$

where, $u_i(\mathbf{x}, t) = \partial h(\mathbf{x}, t)/\partial x_i$, and $f(\mathbf{x}, t)$ is a zero-mean random force described by Eqs. (17) and (18). Assuming homogeneity, we define the generating function by

$$Z(\lambda, \mu_i, t) = \left\langle \exp\left[-i\lambda h(\mathbf{x}, t) - i \sum_{i=1}^{d} = \mu_i \mu_i \right] \right\rangle. \tag{29}$$

As before, λ and μ_i are the sources of $\tilde{h}$ and u_i, respectively, and $i, j = 1, \ldots, d$.

It is then straightforward to derive the following equation for the evolution of $Z(\lambda, \mu_i, t)$:

$$\frac{\partial}{\partial t} z(\lambda, \mu, t) = -\lambda^2 D_0 D(0) Z + \sum_{i=1}^{d} \mu_i^2 D_0 D_{x_i x_i}(0) Z. \tag{30}$$

The joint PDF of h and u_i is obtained by Fourier transform of the generating function:

$$P(h, u, t) = \frac{1}{2\pi} \int d\lambda\, d\mu_i \exp\left(i\lambda h + i \sum_{i=1}^{d} \mu_i u_i \right) Z(\lambda, \mu_i, t). \tag{31}$$

Carrying out the Fourier transformation, we obtain a Fokker-Planck (FP) equation,

$$\frac{\partial}{\partial t} P = D_0 D(0) \frac{\partial^2}{\partial h^2} P - \sum_{i=1}^{d} D_0 D_{x_i x_i}(0) \frac{\partial^2}{\partial u_i^2} P. \tag{32}$$

The solution of the above FP equation is written as, $P(h, \mathbf{u}_i, t) = p_1(h, t) p_2(u_1, t) \ldots p_{d+1}(u_d, t)$(for motivation see Ref. 20). Using the initial conditions that, $p_1(h, 0) = \delta(h)$, and, $p_i(u_i, 0) = \delta(u_i)$, and starting from a flat surface, it can then be shown that,

$$P(h, \mathbf{u}_i, t) = \frac{1}{(2\pi t D_0)^{(d+1)/2} [-D''(0)]^{d/2}} \exp\left[-\frac{h^2}{4 D_0 D(0) t} + \frac{\sum_{i=1}^{d} u_i^2}{4 D_0 D_{xx}(0) t} \right], \tag{33}$$

where, $D'' = D_{x_i x_i} = D_{x_j x_j}$. Thus, the frequency of crossing a definite height, $h(\mathbf{x}, t) = \alpha$, in, for example, the x-direction, is given by,

$$\begin{aligned} \nu_{\alpha x}^{+} &= \int_0^\infty u_x P(\alpha, u_x, u_j)\, du \\ &= \frac{2}{2\pi} \sqrt{-\frac{D_{xx}(0)}{D(0)}} \exp\left[-\frac{\alpha^2}{4 D(0) t} \right] = \frac{1}{2\pi\sigma} \exp\left[-\frac{\alpha^2}{4 D_0 D(0) t} \right],\ j = 2, \ldots, d. \end{aligned} \tag{34}$$

Thus, the quantity $\nu_{\alpha x}^{+}$ in the random deposition model has a Gaussian form with respect to α. Moreover, it is easily seen that $N_x^{+} \sim t^{1/2}$, in agreement with the result for the KPZ equation.

5. SUMMARY

In this paper we derived exact results for the statistics of level-crossing with positive slopes for growing surfaces that are governed by the KPZ equation in $(d+1)$–dimensions with a Gaussian forcing term which is δ–correlated in time and contains short-range spatial correlations, in the limit of zero surface tension. The integral representation of the frequency of crossing, $\nu^{+}_{\alpha x_i}$, in the x_i–direction is given for the KPZ equation in the strong coupling limit before the emergence of sharp valleys (cusp singularities). We also determined the quantity $N^{+}_{x_i} = \int_{-\infty}^{+\infty} d\alpha \nu^{+}_{\alpha x_i}$, which measures the total number of positive-slope crossing of the growing surface in the x_i–direction, and showed that, $N^{+}_{x_i} \sim t^{1/2}$. Using statistical homogeneity, it is then clear that, $\nu^{+}_{\alpha x_i} = \nu^{+}_{\alpha x_j}$, and that, $N^{+} = N^{+}_{x_1} \ldots N^{+}_{x_d} \sim t^{d/2}$. We also derived exact expressions for similar properties in the random deposition model.

The ideas and techniques presented in this paper are quite general and may be used to determine $\nu^{+}_{\alpha x_i}$ for processes that are governed by a general Langevin equation with given drift and diffusion coefficients.

APPENDIX A

In this and the following Appendices we derive the governing equation for the generating function, Eq. (20). In particular, in the present Appendix we investigate a more general definition of the generating function in $(d+1)$–dimensions, and will further expand the study in Appendix B in order to derive an identity which is crucial for deriving Eq. (20) in Appendix C.

We define a generating function by

$$Z(\lambda, \mu_i, \eta_{ij}, x_i, t) = \langle \Theta(\lambda, \mu_i, \eta_{ij}, x_i, t) \rangle, \tag{35}$$

for the fields, $\bar{h} = h - \bar{h}$, $u_i = h_{x_i}$, and $p_{ij} = h_{x_i x_j}$. λ, μ_i, and n_{ij} are the sources of $\tilde{h}$, u_i and p_{ij}, respectively, and $i, j = 1, \ldots, d$. The explicit expression for Θ is given by,

$$\Theta = \exp\left\{ -i\lambda[h(\mathbf{x}, t) - \bar{h}(t)] - i \sum_{i=1}^{d} \mu_i u_i - i \sum_{i \le j = 1}^{d} \eta_{ij} p_{ij} \right\}. \tag{36}$$

We also define q_{ijk} by, $q_{ijk} = h_{x_i x_j x_k}$, which will be used later. Considering the zero-surface tension limit of the KPZ equation. The time evolution of the height $h(\mathbf{x}, t)$ and its derivatives are given by,

$$h_t = \frac{1}{2}\bar{\alpha} \sum_{i=1}^{b} u_i^2 + f, \tag{37}$$

$$u_{i,t} = \bar{\alpha} \sum_{l=1}^{d} u_l p_{li} + f_{x_i}, \tag{38}$$

$$p_{ij,t} = \bar{\alpha} \sum_{l=1}^{d} p_{li} p_{lj} + \bar{\alpha} \sum_{l=1}^{d} u_l q_{lij} + f_{x_i x_j}. \tag{39}$$

Using Novikov's theorem[18,19] (see Appendix D) we can write down the following identities,

$$\langle f \Theta \rangle = -i\lambda k(\mathbf{0}) Z - i \sum_{l=1}^{d} \eta_{ll} k''(\mathbf{0}) Z, \tag{40}$$

$$\langle f_{x_i} \Theta \rangle = -i\mu_i k''(\mathbf{0}) Z, \tag{41}$$

$$\langle f_{x_i x_i} \Theta \rangle = -i\lambda k''(\mathbf{0}) Z - i \sum_{l=1}^{d} \eta_{ll} k''''(\mathbf{0}) Z, \tag{42}$$

$$\langle f_{x_i x_j} \Theta \rangle = -i\eta_{ij} k''''(\mathbf{0}) Z \qquad i \neq j, \tag{43}$$

where, for example, in $d = 3$ we have

$$k(\mathbf{x} - \mathbf{x}') = 2D_0 D(\mathbf{x} - \mathbf{x}'),$$

$$k(\mathbf{0}) = k(\mathbf{0}) = \frac{2D_0}{(\pi\sigma^2)^{3/2}},$$

$$k'(\mathbf{0}) = k_x(\mathbf{0}) = k_y(\mathbf{0}) = k_z(\mathbf{0}) = 0,$$

$$k'' = k_{xx}(\mathbf{0}) = k_{yy}(\mathbf{0}) = k_{zz}(\mathbf{0}) = \frac{-4D_0}{\sigma^2(\pi\sigma^2)^{3/2}}.$$

If we write down the above equations for $(d + 1)$ dimensions, the only change would be having d components for the arguments appearing in k, k'', and k''''.

Differentiating the generating function Z with respect to t, and using Eqs. (37)–(43) and the following identity,

$$iZ_{x_i} - i\lambda Z_{\mu_l} - i \sum_{i} \mu_i Z_{\eta_{il}} \equiv \sum_{i \leq j} \eta_{ij} \langle q_{ijl} \Theta \rangle, \tag{44}$$

we find that the time evolution of Z is governed by

$$Z_t = i\lambda\gamma(t) Z - i\frac{\lambda\bar{\alpha}}{2} \sum_{l} Z_{\mu_l \mu_l} - i\bar{\alpha} \sum_{l} Z_{\eta ll} + i\bar{\alpha} \sum_{l,i \leq j} \eta_{ij} Z_{\eta_{l_i} \eta_{l_i}} - \lambda^2 k(\mathbf{0}) Z$$
$$+ \sum_{l} \mu_l^2 k''(\mathbf{0}) Z - 2\lambda \sum_{l} \eta_{ll} k''(\mathbf{0}) Z - \left(\sum_{l,k} \eta_{ll} \eta_{kk} + \sum_{l<k} \eta_{lk}^2 \right) k''''(\mathbf{0}) Z, \tag{45}$$

where, $\gamma(t) = \bar{h}_t$. If we Fourier transform Z with respect to λ, μ_i, and n_{ij}, we obtain the governing equation for the joint PDF of $\tilde{h}$, u_i, and p_{ij},

$$P(\tilde{h}, u_i, p_{ij}, t) = \int \frac{d\lambda}{2\pi} \prod_i \frac{d\mu_i}{2\pi} \prod_{i \leq j} \frac{d\eta_{ij}}{2\pi} \times \exp\left\{ i\lambda[h(\mathbf{x}, t) + \bar{h}(t)] + i \sum_l \mu_l u_l + i \sum_{l \leq k} \eta_{lk} p_{lk} \right\} \times Z(\lambda, \mu_i, \eta_{ij}, x_i, t). \tag{46}$$

Using Eqs. (45) and (46), the governing equation for the time evolution of $P(\tilde{h}, u_i, p_{ij}, t)$ in d spatial dimensions is then given by,

$$P_t = \gamma(t) P_{\tilde{h}} + \frac{\bar{\alpha}}{2} \sum_l u_l^2 P_{\tilde{h}} - \bar{\alpha}(d+2) \sum_l p_{ll} P - \bar{\alpha} \sum_{l,k \leq m} p_{lk} p_{lm} P_{pkm} + k(\mathbf{0}) P_{\tilde{h}\tilde{h}} - k''(\mathbf{0}) \sum_l P_{u_l u_l}, + 2k''(\mathbf{0}) \sum_l P_{\tilde{h} p_{ll}} + k''''(\mathbf{0}) \sum_{l \leq k} P_{p_{lk} p_{lk}} - 2k''''(\mathbf{0}) \sum_{l < k} P_{p_{ll} p_{kk}}. \tag{47}$$

Equation (47) enables us to obtain the governing equations for the time evolution of the moments of the height and its derivatives. The resulting equation has the following general form,

$$\frac{\partial}{\partial t} \langle \tilde{h}^{n_0} AB \rangle = -n_0 \gamma(t) \langle \tilde{h}^{n_0 - 1} AB \rangle - \frac{\alpha \bar{n}_0}{2} \sum_l \langle \tilde{h}^{n_0 - 1} AB u_1^2 \rangle + \bar{\alpha} \sum_{l,k \leq m} n_{km} \left\langle \tilde{h}^{n_0} AB \frac{p_{lk} p_{lm}}{p_{km}} \right\rangle - \bar{\alpha} \sum_l \langle \tilde{h}^{n_0} AB p_{ll} \rangle + k(\mathbf{0}) n_0 (n_0 - 1) \langle \tilde{h}^{n_0 - 2} AB \rangle - k''(\mathbf{0}) \sum_l n_l (n_l - 1) \left\langle \frac{\tilde{h}^{n_0} AB}{u_l^2} \right\rangle + 2k''(\mathbf{0}) \sum_l n_0 n_{ll} \left\langle \frac{\tilde{h}^{n_0} AB}{p_{ll}^2} \right\rangle + k''''(\mathbf{0}) \sum_{l \leq k} n_{lk} (n_{lk} - 1) \times \left\langle \frac{\tilde{h}^{n_0} AB}{p_{lk}^2} \right\rangle + 2k''''(\mathbf{0}) \sum_{l < k} n_{ll} n_{kk} \left\langle \frac{\tilde{h}^{n_0} AB}{p_{ll} p_{kk}} \right\rangle, \tag{48}$$

where

$$A = \prod_{i=1}^{d} u_i^{n_i},$$

$$B = \prod_{i \le j} p_{ij}^{n_{ij}}.$$

By using various values n_0, n_i, and n_{ij}, coupled differential equations that govern the evolution of the moments are constructed.

APPENDIX B

In this Appendix we prove the identity, $\langle p_{ij} \exp(-i\lambda\tilde{h} - i\sum_l \mu_l u_l)\rangle = 0$. As shown in Appendix C, this identity is crucial to deriving Eq. (20). We also determine the height moments exactly and show that, $\langle h_{x_i x_j}\Theta\rangle = 0$ for $i \neq j$, by considering an initially flat surface. However, it can be shown that this identity also holds for the general $(d+1)$–dimensional surfaces. Setting $n_0 = n_i = n_{ij} = 0$, we obtain from Eq. (48),

$$-\bar{\alpha}\sum_i \langle pu\rangle = -\bar{\alpha}\langle \nabla \cdot \mathbf{u}\rangle = 0 \Rightarrow \langle \nabla \cdot \mathbf{u}\rangle = 0. \tag{49}$$

The main aim is to calculate the moments $\langle p_{ij} \exp(-i\lambda\tilde{h} - i\sum_l u_l u_l)\rangle$. In order to do so, we must follow several steps. First, we must determine such moments as, $\langle p_{ij} u_i u_j\rangle$ for $i \neq j$. Using Eq. (48), we have

$$\begin{aligned}\frac{\partial}{\partial t}\langle u_i u_j p_{ij}\rangle &= \bar{\alpha}\sum_l \langle u_i u_j P_{li} P_{lj}\rangle - \bar{\alpha}\sum_l \langle u_i u_j P_{ll} P_{ij}\rangle \\ &= \bar{\alpha}\sum_l \langle u_i u_j (p_{li} p_{lj} - p_{ll} p_{ij})\rangle.\end{aligned} \tag{50}$$

Inspecting the right-hand side of Eq. (50), we see that the terms with $l = i$ or $l = j$ cancel one another. Noting that, for now, we have restricted our attention to $(3+1)$–dimensions, Eq. (50) is written as

$$\frac{\partial}{\partial t}\langle u_i u_j p_{ij}\rangle = \langle u_i u_j (p_{li} P_{lj} - p_{ll} p_{ij})\rangle \quad i \neq j \neq l. \tag{51}$$

Inserting the right-hand side of Eq. (51) in Eq. (48), one finds

$$\begin{aligned}\frac{\partial}{\partial t}\langle u_i u_j (p_{lj} - p_{ll} p_{ij})\rangle &= \bar{\alpha}\sum_k \langle u_i u_j p_{ki} p_{kl} p_{lj}\rangle + \bar{\alpha}\sum_k \langle u_i u_j p_{kj} p_{kl} p_{il}\rangle \\ &\quad - \bar{\alpha}\sum_k \langle u_i u_j p_{kk} p_{li} p_{lj}\rangle - \bar{\alpha}\sum_k \langle u_i u_j p_{kl} p_{kl} p_{lj}\rangle \\ &\quad - \bar{\alpha}\sum_k \langle u_i u_j p_{ki} p_{kj} p_{ll}\rangle + \bar{\alpha}\sum_k \langle u_i u_j p_{kk} p_{ij} p_{ll}\rangle.\end{aligned} \tag{52}$$

It can then be seen that for $i \neq j \neq l$ the right-hand side of Eq. (52) is zero which, when utilized, yields the following equation for a flat initial surface,

$$\langle u_i u_j (p_{li} p_{lj} - p_{ll} p_{ij}) \rangle = 0. \tag{53}$$

and

$$\langle u_i u_j p_{ij} \rangle = 0. \tag{54}$$

It can also be shown by induction that all the moments $(\tilde{h}^{n_o} p_{ij}^{n_i j} u_i^{ni} u_j^{n_j})$ are identically zero. Therefore, we conclude that, $\langle p_{ij} \exp(-i\lambda\tilde{h} - i \sum_l \mu_l u_l) = 0$.

APPENDIX C

Using the identities that we derived in Appendix B, we derive the joint PDF for the KPZ equation in the limit of zero surface tension. For clarity, we restrict ourselves to the (3 + 1)–dimensional surface, but generalization of the results to any $(d + 1)$–dimensional surface is quite straightforward. The zero-surface tension limit of the KPZ equation in (3 + 1)–dimensions has the following form,

$$h_i(x, y, z, t) - \frac{1}{2}\bar{\alpha}\left(h_x^2 + h_y^2 + h_z^2\right) = f. \tag{55}$$

Letting,

$$h_x = u, \quad h_y = v, \quad h_z = w \tag{56}$$

and differentiating the KPZ equation with respect to x, y, and z, we have

$$\begin{aligned} u_t &= \bar{\alpha}(uu_x + vv_x + ww_x) + f_x, \\ v_t &= \bar{\alpha}(uu_y + vv_y + ww_y) + f_y, \\ w_t &= \bar{\alpha}(uu_z + vv_z + ww_z) + f_z, \end{aligned} \tag{57}$$

for the height h and the corresponding velocity fields. The generating function $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t)$ is defined so as to generate the height and velocity field moments. By defining Θ as

$$\Theta = \exp\{-i\lambda[h(x, y, z, t) - \tilde{h}(t)] - i\mu_1 u - i\mu_2 v - i\mu_3 w\}, \tag{58}$$

the generating function is written as, $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t) = \langle\Theta\rangle$. Using the KPZ equation and its derivatives, the time evolution of $Z(\lambda, \mu_1, \mu_2, \mu_3, x, y, z, t)$ is then written as,

$$\begin{aligned} Z_t = i\gamma(t)\lambda Z - frac12 i\lambda\bar{\alpha}\langle(u^2 + v^2 + w^2)\Theta\rangle - i\bar{\alpha}\mu_1\rangle(uu_x + vv_x + ww_x)\Theta\rangle \\ - i\bar{\alpha}\mu_2\rangle(uu_y + vv_y + ww_y)\Theta\rangle - i\bar{\alpha}\mu_3\langle(uu_z + vv_z + ww_z)\Theta\rangle \\ - i\lambda\langle f\Theta\rangle - i\mu_1\langle f_x\Theta\rangle - i\mu_2\langle f_y\Theta\rangle - i\mu_3\langle f_z\Theta\rangle, \end{aligned} \tag{59}$$

where $\gamma(t) = h_t = \frac{1}{2}\alpha\langle u^2 + v^2 + w^3\rangle$. Utilizing the statistical homogeneity of the quantities, we obtain

$$Z_x = \langle(-i\lambda u - i\mu_1 u_x - i\mu_2 v_x - i\mu_3 w_x)\Theta\rangle, \tag{60}$$

$$Z_y = \langle(-i\lambda v - i\mu_1 v_y - i\mu_2 v_y - i\mu_3 w_y)\Theta\rangle, \tag{61}$$

$$Z_z = \langle(-i\lambda w - i\mu_1 u_z - i\mu_2 v_z - i\mu_3 w_z)\Theta\rangle, \tag{62}$$

Because we analyze the system only in a time regime in which the cusp singularities (sharp valleys) have not yet formed, the order of the partial derivatives can be exchanged,

$$\frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{\partial^2 h}{\partial x_j \partial x_i},$$

and therefore, $u_x = v_y$, $w_x = u_z$, and $w_y = v_z$. Keeping the definition of Θ, Eq. (58), in mind, we may write

$$i\frac{\partial}{\partial \mu_1}\langle(- i\mu_1\mu_x - i\mu_2 v_x - i\mu_3 w_x)\Theta\rangle = \langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uv_x\Theta\rangle - i\mu_3\langle uw_x\Theta\rangle. \tag{63}$$

Using Eqs. (60) and (63) we obtain,

$$\langle u_x\Theta\rangle - i\mu_1\langle uu_x\Theta\rangle - i\mu_2\langle uv_x\Theta\rangle - i\mu_3\langle uw_x\Theta\rangle = -\lambda\frac{\partial}{\partial\mu_1}\langle u\Theta\rangle = -i\lambda Z_{\mu_1\mu_1} \tag{64}$$

In a similar manner we find for the $y-$ and $z-$ directions that,

$$-i\lambda Z_{\mu_2\mu_2} = \langle v_y\Theta\rangle - i\mu_1\langle vu_y\Theta\rangle - i\mu_2\langle vv_y\Theta\rangle - i\mu_3\langle vw_x\Theta\rangle, \tag{65}$$

$$-i\lambda Z_{\mu_3\mu_3} = \langle w_x\Theta\rangle - i\mu_1\langle wu_z\Theta\rangle - i\mu_2\langle wv_z\Theta\rangle - i\mu_3\langle ww_x\Theta\rangle. \tag{66}$$

Using Novikov's theorem[18,19] (see Appendix D) the expressions for $\langle f\Theta\rangle$ and $\langle f_{x_i}\Theta\rangle$ are written in terms of Z (see Appendix A). Therefore, we obtain

$$\begin{aligned} Z_t = {} & i\gamma(t)\lambda Z - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_1\mu_1} - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_2\mu_2} - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_3\mu_3} \\ & - \bar{\alpha}\langle u_x\Theta\rangle - \bar{\alpha}\langle v_y\Theta\rangle - \bar{\alpha}\langle w_z\Theta\rangle - \lambda^2 k(0,0)Z \\ & + \mu_1^2 k_{xx}(\mathbf{0},0)Z + \mu_2^2 k_{xx}(\mathbf{0},0)Z + \mu_3^2 k_{xx}(\mathbf{0},0)Z. \end{aligned} \tag{67}$$

The $\langle u_{x_i}\Theta\rangle$ terms are written as

$$\begin{aligned} \langle u_x\Theta\rangle &= \frac{i}{\mu_1}\langle\Theta\rangle_x + \frac{i}{\mu_1}\langle(i\lambda u + i\mu_2 v_x + i\mu_3 w_x)\Theta\rangle \\ &= -i\frac{\lambda}{\mu_1}Z_{\mu_1} - \frac{\mu_2}{\mu_1}\langle v_x\Theta\rangle - \frac{\mu_3}{\mu_1}\langle w_x\Theta\rangle, \end{aligned} \tag{68}$$

and, similarly, for $\langle v_y\Theta\rangle$ and $\langle w_z\Theta\rangle$ we have

$$\langle v_y\Theta\rangle = -i\frac{\lambda}{\mu_2}Z_{\mu_2} - \frac{\mu_1}{\mu_2}\langle v_y\Theta\rangle - \frac{\mu_3}{\mu_2}\langle w_y\Theta\rangle, \tag{69}$$

$$\langle w_z\Theta\rangle = -i\frac{\lambda}{\mu_3}Z_{\mu_3} - \frac{\mu_1}{\mu_3}\langle u_z\Theta\rangle - \frac{\mu_2}{\mu_3}\langle v_z\Theta\rangle. \tag{70}$$

The terms $\langle h_{x_ix_j}\Theta\rangle(i \neq j)$, that appear in the equation for the time evolution of Z (for example, $\langle u_y\Theta\rangle$), prevent us from writing the Z-equation in a closed form. Fortunately, as was shown in Appendix B, such terms are zero for a flat initial condition. Therefore, the generating function Z satisfies the following equation

$$Z_t = i\gamma(t)\lambda Z - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_1\mu_1} - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_2\mu_2} - \frac{1}{2}i\lambda\bar{\alpha}Z_{\mu_3\mu_3} + i\bar{\alpha}\frac{\lambda}{\mu_1}Z_{\mu_1} - i\bar{\alpha}\frac{\lambda}{\mu_2}Z_{\mu_2}$$
$$-i\bar{\alpha}\frac{\lambda}{\mu_3}Z_{\mu_3} - \lambda^2k(\mathbf{0},0)Z + \mu_1^2k_{xx}(\mathbf{0},0)Z + \mu_2^2k_{xx}(\mathbf{0},0)Z + \mu_3^2k_{xx}(\mathbf{0},0)Z. \tag{71}$$

We now solve Eq. (71) assuming a flat initial condition, $h(x, y, z, 0) = u(x, y, z, 0) = v(x, y, z, 0) = w(x, y, z, 0) = 0$, which is equivalent to writing

$$P(\tilde{h}, u, v, 0) = \delta(\tilde{h})\delta(u)\delta(v)\delta(w), \tag{72}$$

which means that

$$Z(0, \mathbf{0}, t) = 1. \tag{73}$$

An efficient way of solving Eq. (71) is to factorize Z in the following manner,[11,15,16]

$$Z(\lambda, \mu_1, \mu_2, \mu_3, t) = F_1(\lambda, \mu_1, t)F_2(\lambda, \mu_2, t)F_3(\lambda, \mu_3, t)\exp[-\lambda^2k(\mathbf{0})t]. \tag{74}$$

Then, by inserting Eq. (79) in Eq. (71) we obtain

$$F_{1t}F_2F_3 + F_1F_{2t}F_3 + F_1F_2F_{3t} = i\gamma(t)\lambda F_1F_2F_3 - \frac{1}{2}i\lambda\bar{\alpha}F_2F_{1\mu_1\mu_1}F_2F_3$$
$$-\frac{1}{2}i\lambda\bar{\alpha}F_1F_{2\mu_2\mu_2}F_3 - \frac{1}{2}i\lambda\bar{\alpha}F_1F_2F_{3\mu_3\mu_3} + i\bar{\alpha}\frac{\lambda}{\mu_1}F_{1\mu_1}F_2F_3 - i\bar{\alpha}\frac{\lambda}{\mu_2}F_1F_{2\mu_2}F_3$$
$$-i\bar{\alpha}\frac{\lambda}{\mu_3}F_1F_2F_{3\mu_3} - \lambda^2k(\mathbf{0})F_1F_2F_2 + \mu_1^2k''(\mathbf{0})F_1F_2F_3$$
$$+\mu_2k''(\mathbf{0})F_1F_2F_2 + \mu_3k''(\mathbf{0})F_1F_2F_2. \tag{75}$$

Therefore,

$$F_t = -\frac{1}{2}i\lambda\bar{\alpha}F_{\mu\mu} + i\bar{\alpha}\frac{\lambda}{\mu}F_\mu + [\mu^2k''(\mathbf{0}) - i\bar{\alpha}\lambda k''(\mathbf{0})t]F, \tag{76}$$

with the initial condition, $F(\lambda, \mu, 0) = 1$. This indicates that the height gradients in the three dimensions evolve independently of one another before the emergence of cusp singularities, and that they are only coupled with the height field. By Fourier transforming Eq. (76) with respect to μ a simpler partial differential equation of order one is obtained, which can be solved by the method of characteristics.[11] The solution of F_j is then given by

$$F_j(\mu,\lambda,t) = \left\{1 - \tanh^2\left[t\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda}\right]\right\}^{-1/4} \times \exp\left\{-\frac{1}{2}i\bar{\alpha}k''(\mathbf{0})\lambda t^2 - \frac{1}{2}i\mu^2\sqrt{\frac{2ik_{xx}(\mathbf{0})}{\bar{\alpha}}}\tanh\left[t\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda}\right]\right\}. \quad (77)$$

Therefore, we obtain the generating function in (3 + 1)–dimensions,

$$Z(\lambda,\mu_1,\mu_2,t) = \left\{1 - \tanh^2\left[t\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda}\right]\right\}^{-3/4} \times \exp\left\{-\frac{1}{2}i\left(\mu_1^2+\mu_2^2+\mu_3^2\right)\sqrt{\frac{2ik_{xx}(\mathbf{0})}{\bar{\alpha}\lambda}}\tanh\left[t\sqrt{2ik_{xx}(\mathbf{0})\bar{\alpha}\lambda}\right]\right\} \times \exp\left[-\frac{3}{2}i\bar{\alpha}k''(\mathbf{0})\lambda t^2 - k(\mathbf{0})\lambda^2 t\right]. \quad (78)$$

By inverse Fourier transformation of the generating function Z, the PDF of the height fluctuations, $P(\tilde{h},u,v,w,t)$, is then determined.

As mentioned earlier, generalizing the above results to any $(d+1)$–dimensional surface is straight forward. For example, for $d=2$ all we need to do is setting $\mu_3 = 0$ in Eq. (78). For a general d we have the following expression for Z:

$$Z(\lambda,\mu_1\cdots,\mu_d,t) = F_1(\lambda,\mu_1,t)\cdots F_d(\lambda,\mu_d,t)\exp[-\lambda^2 k(\mathbf{0})t], \quad (79)$$

where F_j is defined as before.

APPENDIX D

In this Appendix we prove a simple version of the theorem by Novikov[18,19] that we use in this paper. As before, we define the generating, $Z(\lambda,\mu_i,x_i,t) = \langle\Theta(\lambda,\mu_i,x_i,t)\rangle$, for the fields $\tilde{h} = h(\mathbf{x},t) - \bar{h}$ and $u_i = h_{x_i} = \partial h/\partial x_i$ by

$$\Theta = \exp\left\{-i\lambda[h(x,y,z,t) - \bar{h}(t)] - i\sum_{i=1}^{d}\mu_i u_i\right\}, \quad (80)$$

where, as usual, λ and μ_i are the sources of $\tilde{h}$ and u_i, respectively, and $i,j = 1,\cdots,d$. Starting with the equation,

$$\partial_t h(\mathbf{x},t) = -\mathcal{L}[h(\boldsymbol{x},t)] + f(\boldsymbol{x},t) \quad (81)$$

where $f(\mathbf{x}, t)$ is a term representing the noise with Gaussian correlations in space and white noise in time:

$$\langle f(\mathbf{x}, t) f(\mathbf{x}', t')\rangle = 2D_0 D(\mathbf{x} - \mathbf{x}')\delta(t - t'), \tag{82}$$

we can write,

$$\langle F[f] f(\mathbf{x}, t)\rangle = \int d\mathbf{x}'\, dt' \langle f(\mathbf{x}, t) f(\mathbf{x}', t')\rangle \left\langle \frac{\partial F[f]}{\partial f(\mathbf{x}', t')}\right\rangle \tag{83}$$

If we take $F = \Theta$, we obtain

$$\frac{\partial F[f]}{\partial f(\mathbf{x}', t')} = -i\lambda \frac{\partial h(\mathbf{x}, t)}{\partial f(\mathbf{x}', t')} F[f] - i \sum_{i=1}^{d} \mu_i \frac{\partial u_i(\mathbf{x}, t)}{\partial f(\mathbf{x}', t')} F[f]. \tag{84}$$

Using Eq. (81) we can write,

$$h(\mathbf{x}, t) = h(\mathbf{x}, t_0) - \int_{t_0}^{t} \mathcal{L}[h(\mathbf{x}, t)]\, dt' + \int_{t_0}^{t} f(\mathbf{x}, t')\, dt' \tag{85}$$

which, when differentiated with respect to f, gives,

$$\frac{\partial h(\mathbf{x}, t)}{\partial f(\mathbf{x}, t)} = -\int_{t_0}^{t} \frac{\partial \mathcal{L}[h(\mathbf{x}, t)]}{f(\mathbf{x}, t)}\, dt' + \delta(\mathbf{x} - \mathbf{x}'). \tag{86}$$

Using Eqs. (84) and (86) in the limit, $t - t' \to 0$, invoking casuality, and considering the same expressions for, $\sum_{i=1}^{d} \mu_i \partial u_i(\mathbf{x}, t)/\partial f(\mathbf{x}', t') F[f]$, we obtain

$$\langle F[f] f(\mathbf{x}, t)\rangle = -2i\lambda D_0 D(0)\langle F[f]\rangle - 2i \sum_{i=1}^{d} \mu_i D_0 D''(\mathbf{x} - \mathbf{x}')\langle F[f]\rangle. \tag{87}$$

Using the last expressions and $k(\mathbf{x}) = 2D_0 D(\mathbf{x})$, we obtain Eq. (40).

ACKNOWLEDGMENT

We would like to thank M. Reza Rahimi Tabar for useful comments.

REFERENCES

1. A.-L. Barabási and H. E. Stanley, *Fractal Concepts in Surface Growth*. (Cambridge University Press, New York, 1995).
2. T. Halpin-Healy and Y. C. Zhang, *Phys. Rep.* **245:** 218 (1995); J. Krug, *Adv. Phys.* **46**:139 (1997).
3. J. Krug and H. Spohn, In C. Godreche, editor, *Solids Far from Equilibrium Growth, Morphology and Defects*. (Cambridge University Press, New York, 1990).
4. P. Meakin, *Fractal, Scaling and Growth Far from Equilibrium*. (Cambridge University Press, Cambridge, 1998).
5. M. Marsilli, A. Maritan, F. Toigoend and J. R. Banavar, *Rev. Mod. Phys.* **68**:963 (1996).
6. M. Kardar, *Physica A* **281**:295 (2000).

7. F. Family and T. Vicsek, *J. Phys. A* **18**:L75 (1985).
8. J. Krug, H. Kallabis, S. N. Majumdar, S. J. Cornell, A. J. Bray and C. Sire, *Phys. Rev. E* **56**:2702 (1997).
9. For a review see, for example, S.N. Majumdar, *Curr. Sci.* **77**:370 (1999).
10. S. N. Majumdar and A. J. Bray, *Phys. Rev. Lett.* **86**:3700 (2001).
11. S. O. Rice, *Bell System Tech. J.* **23**:282 (1944); **24**:46 (1945).
12. D. E. Newland, *An Introduction to Random Vibration, Spectral and Wavelet Analysis*. (Longman, Harlow and Wiley, New York, 1993).
13. A. A. Masoudi, F. Shahbazi, J. Davoudi and M. R. Rahimi Tabar, *Phys. Rev. E* **65**:026132 (2002).
14. F. Shahbazi, S. Sobhanian, M. R. Rahimi Tabar, S. Khorram, G. R. Frootan and H. Zahed, *J. Phys. A* **36**:2517 (2003).
15. A. Bahraminasab, S. M. A. Tabei, A. A. Masoudi, F. Shahbazi and M. R. Rahimi Tabar, *J. Stat. Phys.* **116**:1521 (2004).
16. S. M. A. Tabei, A. Bahraminasab, A. A. Masoudi, S. S. Mousavi and M. R. Rahimi Tabar, *Phys. Rev. E* **70**:031101 (2004).
17. R. Piessens, E. de Doncker, C. Uberhuber and D. K. Kahaner, *QUADPACK-A Subroutine Package for Automatic Integration*. Springer Series in Computational Mathematics (Springer, Berlin, 1983).
18. E. A. Novikov, *Sov. Phys. JETP* **20**:1290 (1965).
19. J. Garc̀ýa-Ojalvo and J. M. Sancho, *Noise in Spatially Extended Systems*. (Springer-Verlag, New York, 1999).
20. K. R. Sreenivasan, A. Prabhu and R. Narasimha, *J. Fluid Mech.* **137**:251 (1983).